Almost Classical Skew Bracoids

Isabel Martin-Lyons

and Paul Truman

Keele University, UK

Hopf algebras and Galois module theory

Tuesday 30^{th} May, 2023

Motivation

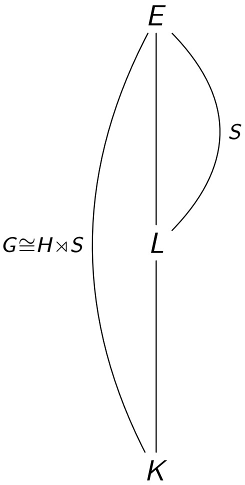
Let L/K be a separable extension of fields with Galois closure E, and write $G = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$. Recall:

Definition

We say that L/K is *almost classically Galois* if S has a normal complement H in G.

Definition

A Hopf-Galois structure is *almost classical* if it corresponds under Greither-Pareigis to a subgroup of $\mathrm{Perm}(G/S)$ of the form $\lambda(H)^{opp}$, for some normal complement H of S.



Recap of the correspondence

Recall that a skew bracoid corresponding to a Hopf-Galois structure on a separable extension L/K looks like $(G, \cdot, G/S, \star, \odot)$ where:

- $(G, \cdot) = \mathrm{Gal}(E/K)$,
- G acts on G/S via left translation of cosets, denoted $\odot$,
- the binary operation $\star$ is manufactured so that $(G/S, \star)$ is a group and the relation

$$g \odot (\bar{x} \star \bar{y}) = (g \odot \bar{x}) \star (g \odot \bar{e})^{-1} \star (g \odot \bar{y})$$

  holds for all $g \in G$ and $\bar{x}, \bar{y} \in G/S$.

Note that $S = \mathrm{Stab}_G(\bar{e})$ under this action.

What's this to a skew bracoid?

Immediately then, if we come from L/K almost classically Galois then $(G, \cdot) \cong H \rtimes S$ for some $H \trianglelefteq G$.

Proposition

With this set-up $(H, G/S)$ is a sub-skew bracoid of $(G, G/S)$ and is essentially a skew brace.

Proof.

- Certainly $(H, \cdot) \leq (G, \cdot)$ and $(G/S, \star) \leq (G/S, \star)$.
- $G/S = G \odot \bar{e} = (H \cdot S) \odot \bar{e} = H \odot \bar{e}$ so H acts transitively.
- $|G| = |H||S|$ as $G = HS$ and $|G| = |G/S||S|$ by Orbit Stabiliser so $|H| = |G/S|$ and $(H, G/S)$ is essentially a skew brace.

$\square$

To simplify things

It is going to be helpful for us to fix H as a system of representatives for the coset space G/S. If we do this then the action of $h_1x_1 \in G$ on $\bar{h_2} \in G/S$ by left translation of cosets is written

$$\begin{aligned} h_1x_1 \odot h_2S &= h_1x_1h_2S \\ &= h_1x_1h_2x_1^{-1}S, \end{aligned}$$

conjugation the S component and left multiplication by the H component.

And the structures?

Let's take a closer look at $\lambda(H)^{opp}$.

Proposition

The map $\rho_H : H \to \mathrm{Perm}(G/S)$ given by $\rho_H(h_1)[\bar{h_2}] = \overline{h_2 h_1^{-1}}$ for all $h_1, h_2 \in H$ is well defined and its image coincides with $\lambda(H)^{opp}$.

Proof.

Since each coset in G/S has a unique representative in H, the permutation is completely (well) defined. Then

$$\lambda(H)^{opp} = \{\eta \in \mathrm{Perm}(G/S) : \lambda(h)\eta = \eta\lambda(h) \text{ for all } h \in H\},$$

and $\lambda(h_1)\rho_H(h_2)h_3S = (h_1 h_3 h_2^{-1})S = \rho_H(h_2)\lambda(h_1)h_3S$ for all $h_i \in H$. Hence, $\rho_H(H) \subseteq \lambda(H)^{opp}$ and a size argument gives equality. $\square$

And the structures?

Recall we define $\bar{x} \star \bar{y} := a(a^{-1}(\bar{x})a^{-1}(\bar{y}))$ in G/S, where $a : \eta \mapsto \eta^{-1}[\bar{e}]$.

For us $a : \rho_H(H) \to G/S$ is then $a(\rho_H(h)) = \rho_H(h)^{-1}[\bar{e}] = hS$.
Now for $h_1, h_2 \in H$, we have

$$\begin{aligned} \overline{h}_1 \star \overline{h}_2 &:= a(a^{-1}(\overline{h}_1)a^{-1}(\overline{h}_2)) \\ &= a(\rho(h_1)\rho(h_2)) \\ &= \rho(h_2)^{-1}\rho(h_1)^{-1}[\bar{e}] \\ &= \rho(h_2)^{-1}[\overline{h}_1] \\ &= \overline{h_1 h_2}, \end{aligned}$$

so that the star operation is really the dot operation in H.

In the abstract

To emphasise, for all $h_1, h_2 \in H$ we have

$$\bar{h}_1 \star \bar{h}_2 = \overline{h_1 h_2} = h_1 h_2 \odot \bar{e} = h_1 \odot \bar{h}_2,$$

so that $(h_1 \odot \bar{e})^{-1} = h_1^{-1} \odot \bar{e}$.

Hence, in the almost classical structures, our sub-skew bracoid $(H, \cdot, G/S, \star, \odot)$ of $(G, \cdot, G/S, \star, \odot)$ is not only essentially a skew brace, but essentially a *trivial* skew brace. With this in mind...

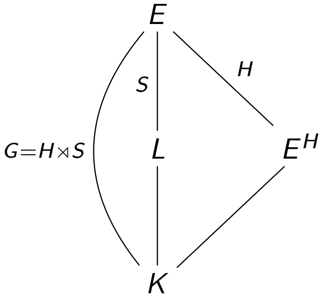
Definition

Let (G, N) be a skew bracoid and $S = \mathrm{Stab}_G(e_N)$. We say (G, N) is *almost classical* if S has a normal complement H in G for which (H, N) is essentially a trivial skew brace.

Back to the lattice

Remark

Note that under the new Stefanello-Trappeniers correspondence, this $(H, G/S)$ would line up with the classical structure on some relevant Galois extension. For example, E/E^H in our original set-up.



The γ-functions

In terms of the gamma function, if we have an almost classical skew bracoid (G, N) and we let $hx \in G \cong H \rtimes S$ and $n = h_n \odot e_N \in N$ then,

$$\begin{aligned} {}^{\gamma(hx)}n &= (hx \odot e_N)^{-1} \star (hx \odot n) \\ &= (h \odot e_N)^{-1} \star (hxh_n \odot e_N) \\ &= (h^{-1} \odot e_N) \star (hxh_n x^{-1} \odot e_N) \\ &= xh_n x^{-1} \odot e_N. \end{aligned}$$

So we conjugate by the S part and the H part acts trivially.

Surjectivity of the Hopf-Galois Correspondence

By classical Galois theory the intermediate fields of L/K line up with subgroups G' of G that contain S. These intermediate fields then show up in the image of the HGC if and only if $G' \odot \bar{e}$ is a left ideal of $(G, G/S)$.

Theorem

If $(G, G/S)$ is an almost classical skew bracoid then $G' \odot \bar{e}$ is a left ideal for all G' with $S \leq G' \leq G$.

Proof.

Let $(G, G/S)$ be an almost classical skew bracoid and G' be a subgroup of G containing S. Let $hx \in G$ and $h'x' \in G'$, then

$$^{\gamma(hx)}(h'x' \odot \bar{e}) = {}^{\gamma(hx)}(h' \odot \bar{e}) = xh'x^{-1} \odot \bar{e}.$$

This is in $G' \odot \bar{e}$, as $x, x^{-1} \in S \subseteq G'$ and $h' = h'x'(x')^{-1} \in G'$. $\square$

To the holomorph!

To move from a skew bracoid (G, N) to a subgroup of $\mathrm{Hol}(N)$ we use the map $\Gamma : G \to \mathrm{Hol}(N)$ given by $g \mapsto (g \odot e_N, \gamma(g))$.

Proposition

If (G, N) is an almost classical skew bracoid then $(N, id) \subseteq \Gamma(G)$.

Proof.

Let H be a relevant normal complement to $S = \mathrm{Stab}_G(e_N)$ in G. Then,

$$\begin{aligned}\Gamma(H) &= \{(h \odot e_N, \gamma(h)) | h \in H\} \\ &= \{(h \odot e_N, id) | h \in H\} \\ &= (N, id).\end{aligned}$$

$\square$

And back

Note also that if $g \mapsto (e_N, \gamma(g))$ then in particular $g \odot e_N = e_N$. Hence, $\Gamma(S)$ is precisely the automorphism part of $\Gamma(G)$.

Suppose conversely that we have a transitive subgroup of $\mathrm{Hol}(N)$ of the form $G = N \rtimes A$ for some $A \subseteq \mathrm{Aut}(N)$. Thought of as a skew bracoid (G, N) we have

- (N, N) as an essentially trivial sub-skew bracoid,
- and A, as the automorphism part of G, is $\mathrm{Stab}_G(e_N)$,

i.e. we have an almost classical skew bracoid.

Induced Hopf-Galois Structures

$$
\begin{tikzcd}
E \arrow[d, dash, "\begin{matrix} S \\ N \end{matrix}"', "{(S, \star_N, \cdot)}"] \arrow[dd, dash, bend right=60, "\begin{matrix} (G,\cdot) \cong H \rtimes S \\ M \times N \\ (G, G/S \times S) \end{matrix}"'] \\
L \arrow[d, dash, "M"', "{(G, G/S)}"] \\
K
\end{tikzcd}
$$

Here $N \subseteq \mathrm{Perm}(S)$ and $M \subseteq \mathrm{Perm}(G/S)$, leading to a subgroup of $\mathrm{Perm}(G)$ isomorphic to $M \times N$.

Taking a direct product of the star groups of $(G, \cdot, G/S, \star_M, \odot)$ and $(S, \star_N, \cdot)$ we can construct a skew bracoid $(G, G/S \times S)$ on E/K, which is essentially a skew brace.

The Induced skew brace

Note that as $(G, \cdot) \cong H \rtimes S$ the composite map $G/S \times S \to H \times S \to G$ given by $(\bar{h}, x) \mapsto (h, x) \mapsto hx$ is a well defined bijection.

Then, the action of $hx \in G$ on $(\overline{h'}, x') \in G/S \times S$ is explicitly given by

$$hx(\overline{h'}, x') = (\overline{hxh'x^{-1}}, xx') \mapsto hxh'x^{-1}xx' = hxh'x',$$

coinciding (as we would hope) with the original operation in G.

Definition (Tentatively)

We say that a skew brace $(G, \star, \cdot)$ is *induced* if it contains some sub-skew braces $(H, \star, \cdot)$ and $(S, \star, \cdot)$ such that $(G, \cdot) \cong (H, \cdot) \rtimes (S, \cdot)$ and $(G, \star) \cong (H, \star) \times (S, \star)$.

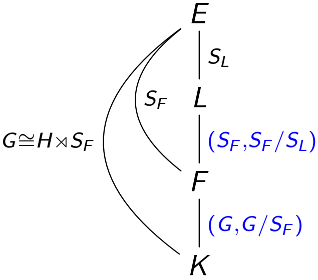
Remarks

Remark

If we use the trivial skew brace on E/L and the almost classical skew bracoid on $E/L/K$ then we obtain the skew brac(oid) $(H \rtimes S, H \times S)$.

Remark

Everything "works" if we follow this construction with two skew bracoids, yielding a skew bracoid. This may guide a theory of induced Hopf-Galois structures on separable extensions.



Where next?

- See how this lines up with existing classifications, for example [Koh98], [Byo07], [CS19],[CS20],[CS21] and apply this to for example [Dar23].
- There must be a short exact sequence at play, if we give ourselves all of the induced tools we have for example:

$$e \to (S, \star, \cdot) \to (G, \star, \cdot) \to (G, G/S) \to e.$$

- Could a similar construction be devised that doesn't always give a direct product in the additive part? Perhaps one that lands on a trivial skew brace when applied to a trivial skew brace and an almost classical skew bracoid?

Thank you for your attention!